Economic Growth

ECON 302, Intermediate Macroeconomics

Nicolas Franz-Pattillo

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University of British Columbia

This set of slides

Introduction

Introduction

- In general, we think about "economic development" as an increase in "living standards" (vague concept)
- Economic growth is not the same than economic development, but most likely correlated (see Jones and Klenow, 2016)
- Most studies use GDP per capita to account for economic growth: $y_t \equiv \dfrac{Y_t}{N_t}$ why?
- We saw in the previous lecture that some countries GDP grew significantly during the XX century
- Short vs Long-run.

Introduction: goals

- At the end of this lecture you should be able to understand what a production function is and how its factors contribute to GDP growth.
- You should be capable to understand that long run growth is a bit more complicated than just saving more or accumulating capital faster.
- Similarly, the differences in GDP per-capita today, might be a consequence of differences in the “way of doing things” rather than the amount of inputs we use.

This class contents can be found in:

- Mankiw (2009, chapters 3.1 and 7)
- Abel et al. (2018, chapters 3.1 and 6)

If you find this topic interesting, you might want to check

- Acemoglu (2007, ch.1-3) *strongly recommended*
- Barro and Sala-i Martin (2003, ch.1)
- Caselli (2005)
- Jones (2016)
- Jones and Romer (2010)
- Solow (1999)

QUESTIONS?

Short math review

Short math review

- Assume that the variable X_t grows proportionally at rate g

$$\Delta\% X_{t+1} = \frac{\Delta X_{t+1}}{X_t} = \frac{X_{t+1} - X_t}{X_t} = \frac{X_{t+1}}{X_t} - 1 = g \tag{1}$$

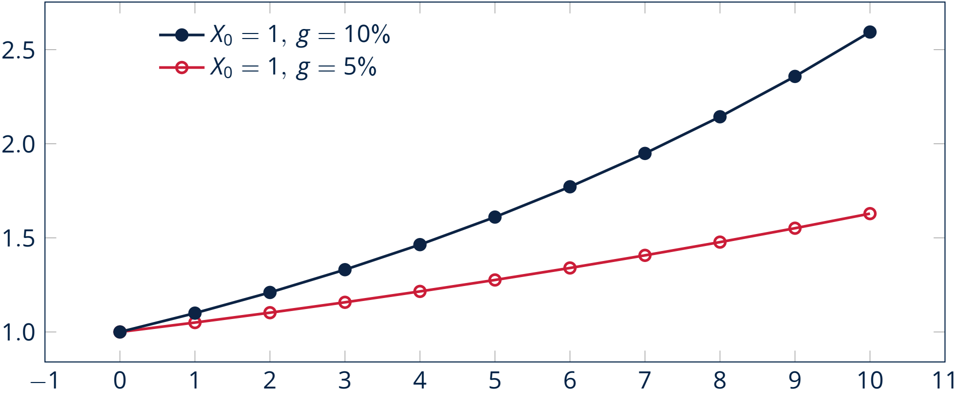
- Reorganizing the previous equation we get

$$X_{t+1} = (1+g)X_t$$

- More generally

$$X_{t+j} = (1+g)^{j} X_t \tag{2}$$

Figure 1: Proportional growth of X



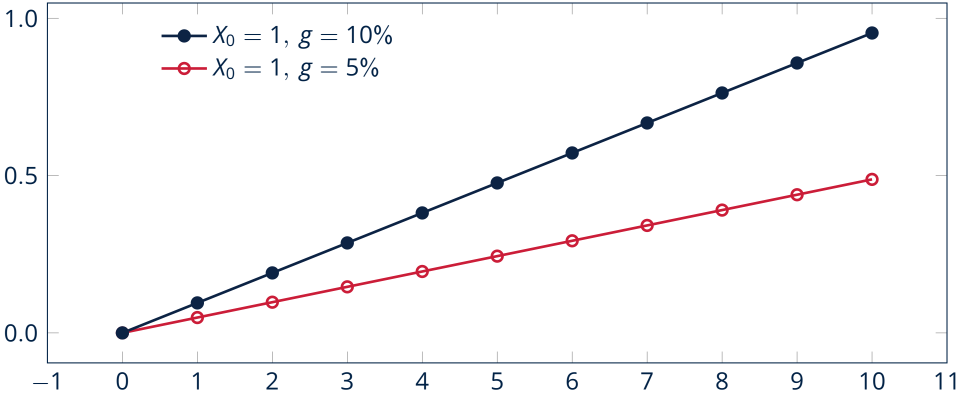
Proportional and Log-linear growth

When the variable X_t growths proportionally at rate g , with g small, then the $\log(X_t)$ grows linearly over time with slope g

Proof: take equation (2) in logs and note that $\Delta t = j$

$$\begin{aligned} \log\left(X_{t+j}\right) - \log\left(X_t\right) &= \log\left[(1+g)^j\right] \\ \implies \frac{\Delta \log\left(X\right)}{\Delta t} &= \log(1+g) \\ \implies \frac{\Delta \log\left(X\right)}{\Delta t} &\approx g \quad \square \end{aligned}$$

Figure 2: Linear growth of $\log(X)$



Proportional and Log-linear growth 2.0

If the variables X_t and Z_t both grow at a proportional rate g, the difference $X_t - Z_t$ will grow but $\log(X_t) - \log(Z_t)$ will remain constant

Proof: for the first statement we know by (2) that

$$
\begin{aligned}
X_t - Z_t &= (1+g)X_{t-1} - (1+g)Z_{t-1} \\
\Delta\%\left[X_t - Z_t\right] &= \frac{X_t - Z_t}{X_{t-1} - Z_{t-1}} - 1 = g \quad \square
\end{aligned}
$$

For the second statement we use the fact that

$$
\begin{aligned}
\log(X_t) - \log(Z_t) &= \log\left(\frac{X_t}{Z_t}\right) = \log\left(\frac{\cancel{1+g}}{\cancel{1+g}}\frac{X_{t-1}}{Z_{t-1}}\right) \\
\implies \log(X_t) - \log(Z_t) &= \log(X_{t-1}) - \log(Z_{t-1}) \quad \square
\end{aligned}
$$

- Assume the growth rate of X changes over time (g_t). What is the average growth rate?
- What about the arithmetic average: $\frac{1}{T}\sum_{t=1}^{T} g_t$

Average of growth rates

t	X_t	g_t (%)
1	100.0	-
2	50.0	-50%
3	75.0	50%
4	79.5	6%

- We start with 100, end with 79.5 and the average of the growth rates is 2%
- Is this average informative?

- What about the geometric average?

$$1 + \bar{g} = \Pi_{i=1}^{j}(1 + g_{t+i})^{1/j}$$

- Use this definition and (2) to get

$$\bar{g} = \left(\frac{X_{t+j}}{X_t}\right)^{1/j} - 1 \tag{3}$$

Examples

1. Whats the annual average growth rate if $X_1 = 100$ and $X_4 = 79.5$? -7.36%
2. How long will it take to get from $10,000$ USD per-capita to $20,000$ USD per-capita if the country grows annually at 4%? 17.7 years

- Discrete vs Continuous time: is there any difference?
- We know that in discrete time a variable that grows proportionately is described by:

$$X_t = (1+g)^t X_0$$

- What about in continuous time?

$$X(t) = X(0) \exp\{g \cdot t\}$$

- In continuous time the "instantaneous" growth rate is

$$\frac{d\ln[X(t)]}{dt} = \frac{1}{X(t)} \cdot \frac{dX(t)}{dt} = \frac{\dot{X}(t)}{X(t)} = g \tag{4}$$

Notation

- During this course we will use the following notation for the partial derivatives

$$\frac{\partial F(X,Y)}{\partial X} \equiv F_X(X,Y)$$

- When the function is univariate we might use

$$\frac{\partial F(X)}{\partial X} \equiv F'(X)$$

- Finally, when a variable changes over time we will use

$$\frac{dX(t)}{dt} = \dot{X}(t)$$

QUESTIONS?

Data and Facts

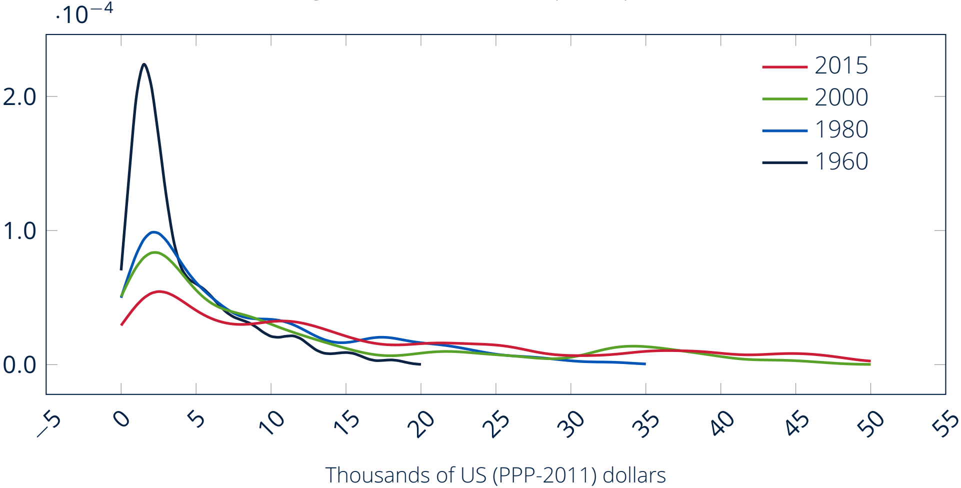
Data and Facts: Kaldor

Kaldor (1961) highlighted six “stylized” facts to summarize the patterns of economic growth

1. Labor productivity has grown at a sustained rate.
2. Capital per worker has also grown at a sustained rate.
3. The real interest rate, or return on capital, has been stable.
4. The ratio of capital to output has also been stable.
5. Capital and labor have captured stable shares of national income.

Data and Facts: GDP varies greatly around the world

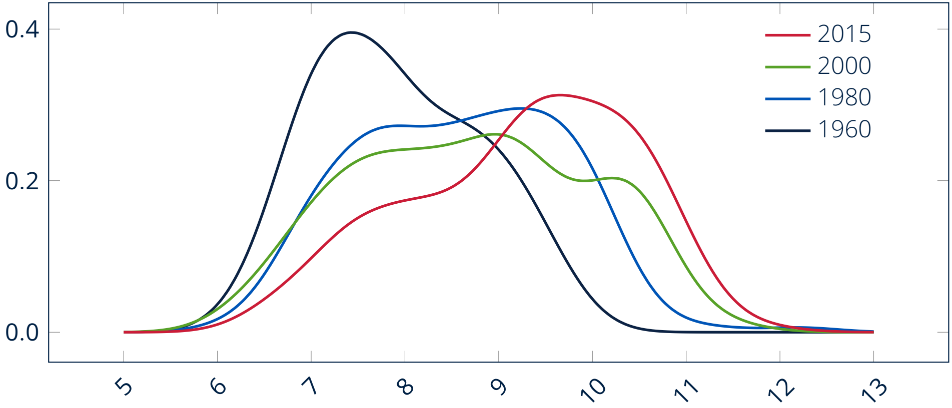
Figure 3: Distribution of per-capita GDP



Source: Data from Maddison Project Database, version 2018, and own calculations.

- It looks like the cross-country distribution of GDP is spreading.
- If countries grow, is the previous graph a good representation of the story?
  - Even if countries grow at the same rate but start from different levels, then the distribution will spread
  - How do we account for this? look at the math review

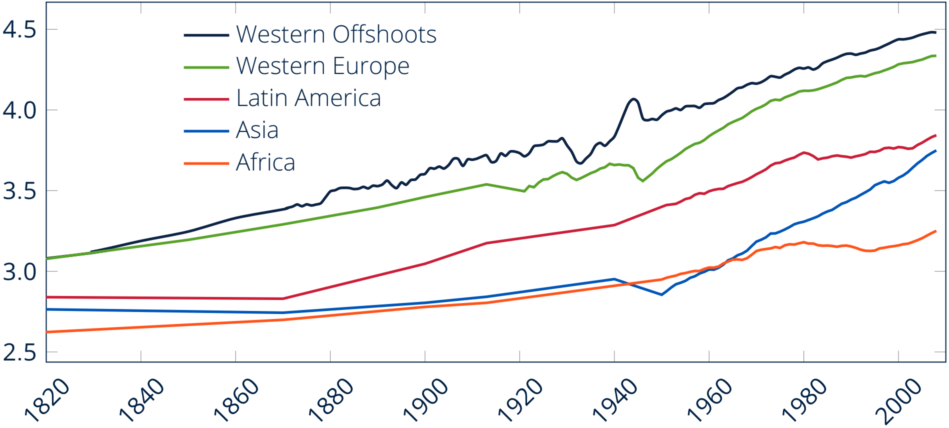
Figure 4: Distribution of per-capita GDP (Logs)



Source: Data from Maddison Project Database, version 2018 and own calculations.

Data and Facts: growth rates are not the same

Figure 5: GDP per-capita (Logs, selected regions)



Source: Maddison Project Database, version 2013

Data and Facts: growth rates are not the same

- Some regions are growing at the same pace.
- Some of them are catching up
- Others are lagging behind

Jones and Romer (2010) update the Kaldor facts

1. Increases in the extent of the market. Increased flows of goods, ideas, finance, and people have increased the extent of the market for all workers and consumers
2. Accelerating growth. Growth in both population and per capita GDP has accelerated, rising from virtually zero to the relatively rapid rates observed in the last century
3. Variation in modern growth rates. The variation in the rate of growth of per capita GDP increases with the distance from the technology frontier.

Data and Facts: Jones and Romer (cont.)

4. Large income and total factor productivity (TFP) differences. Differences in measured inputs explain less than half of the enormous cross-country differences in per capita GDP.

5. Increases in human capital per worker. Human capital per worker is rising dramatically throughout the world.

6. Long-run stability of relative wages.The rising quantity of human capital, relative to unskilled labor, has not been matched by a sustained decline in its relative price.

Data and Facts: Reminder

Remember that economic development in not the same than economic growth. Moreover, economic growth involves costs that are difficult to measure or weight:

- Short and medium run costs.
- Growth and inequality. causality?
- Does industrialization damage the environment? how do we measure this cost?
- “Labor standards” and foregone leisure.

The simple model we will look at ignore these questions, but that does not mean that they are not relevant.

QUESTIONS?

The production function

How is output produced?

- GDP is produced using many factors: machines, labor, energy, land, raw materials, and more
- We focus on two factors: Capital (K) and Labor (N)
- An important concept is **productivity**: the effectiveness of the productive effort
  - Reflects the idea that more productive economies produce more with the same levels of K and N
  - Depends on technology, management, efficient allocation of resources, human capital, other institutions, etc.

A production function transforms any given level of capital, labor, and productivity into output:

$$Y_t = A_t F(K_t, N_t) \tag{5}$$

- Y_t is the aggregate physical output produced during period t (month, quarter, year...)
- K_t is the stock of capital available during period t
- N_t is the labor input (or total hours of work) during period t
- A_t is Total Factor Productivity (**TFP**) - a variable that captures productivity elements.
- $F(\cdot, \cdot)$ is a time-invariant function.

The production function

We will assume that the Production Function have the following properties

1. Positive Marginal Product (more input $\implies$ more output)

$$MPK = F_K > 0 \; ; \qquad MPN = F_N > 0$$

2. Decreasing Marginal Products (the last unit of input added is less productive than the previous)

$$F_{KK} < 0 \; ; \qquad F_{NN} < 0$$

3. Complementarity (i.e. more labor $\implies$ higher marginal product of capital and vice-versa)

$$F_{NK} > 0 \; ; \qquad F_{KN} > 0$$

4. Constant returns to scale (double the inputs $\implies$ double the output)

$$AF(\lambda K, \lambda N) = A\lambda F(K, N)$$

Figure 6: Production Function

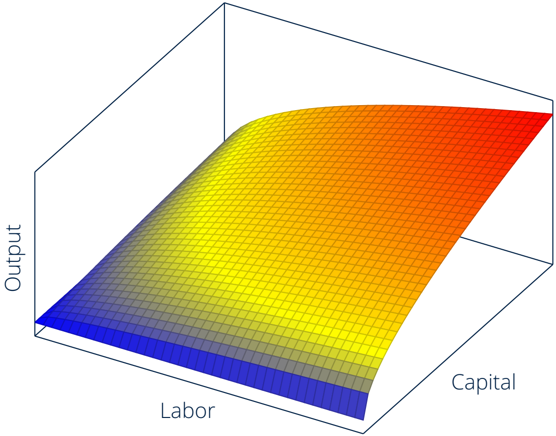
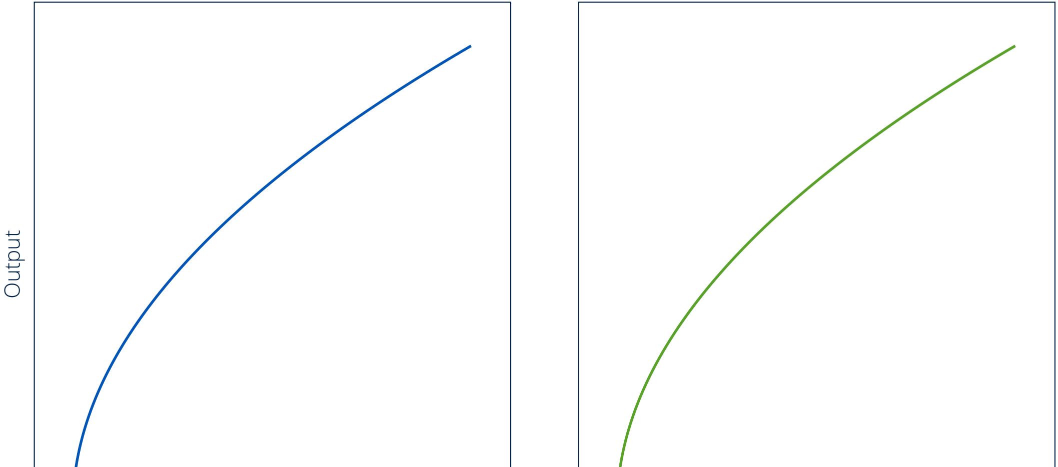


Figure 7: Production Function

(a) Fixed labor: $F\left(K, \overline{L}\right)$

(b) Fixed Capital: $F\left(\overline{K}, L\right)$



The production function: Example

Cobb-Douglas production function

$$Y_t = A_t K_t^{\alpha} N_t^{\beta} \tag{6}$$

with $\alpha \in (0,1)$ and $\beta \in (0,1)$

1. Heavily used...
2. Satisfies the four properties? only if $\alpha + \beta = 1$
3. Elasticities: (Δ% in output with small Δ% in input)
   - α is the elasticity of output w.r.t. Capital
   - β is the elasticity of output w.r.t. Labor
4. Income shares (Payments to the input as % of the total)
   - α is the Capital share of income
   - β is the Labor share of income

The production function: Example

Cobb-Douglas production function

- The proof of point 2 is trivial.
- To prove point 3 we just need to realize that the output elasticity of Capital is

$$\epsilon_{Y,K} = \frac{dY}{dK}\frac{K}{Y} = \frac{d\ln Y}{d\ln K}$$

Using this with the Cobb-Douglas production function we get

$$\frac{d\ln Y}{d\ln K} = \frac{\ln A_t + \alpha \ln K_t + \beta \ln N_t}{d\ln K} = \alpha$$

- The proof of point 4 requires a bit more elaboration on what is the equilibrium income share and, therefore, the firm's problem.

The Firms' Problem

Competitive Markets

Implies that no economic agent can influence the price of inputs or outputs alone. It requires one or more of the following conditions:

1. All agents sufficiently small.
2. Homogeneity of inputs/outputs, and
3. Frictionless reallocation

A non-competitive market example would be **monopolistic competition**

The Firms' Problem: Assumptions

We will assume the following:

- The firm uses the technology $F(K, N)$ to produce, and sells its output at price price P
  (Normalize $P = 1$ and all other variables become real variables)
- Hires labor, N, at price w per unit
- Rents capital, K, at price r per unit
- Its sole objective is to maximize profits

The Firm Problem: Solution

Profit maximization

$$\max_{\{K,N\}} A_t F(K_t, N_t) - w_t N_t - r_t K_t \tag{7}$$

The first order conditions are

$$r_t = A_t F_K(K_t, N_t) \tag{8}$$

$$w_t = A_t F_N(K_t, N_t) \tag{9}$$

Factors are paid their marginal products!

Income share of Capital and Labor

The share of income of capital is:

$$\frac{\text{Capital Rent} \times \text{Capital}}{\text{Income}} = \frac{r \times K}{Y}$$

Similarly, the share of labor is

$$\frac{\text{Wage} \times \text{Hours Worked}}{\text{Income}} = \frac{w \times N}{Y}$$

These shares can be computed using GDP series (income approach)

Income shares of Capital: Cobb-Douglas

Replace $r = A_t F_K(K_t, N_t)$ from the solution of the Firm's problem, and $F(K, N) = K_t^\alpha N_t^\beta$ to get

$$
\begin{aligned}
\frac{r \times K}{Y} &= \frac{F_K(K_t, N_t) K_t}{F(K_t, N_t)} \\
&= \alpha \frac{K_t^{\alpha-1} N_t^\beta K_t}{K_t^\alpha N_t^\beta} \\
&= \alpha \quad \square
\end{aligned}
$$

- This is useful because we have data on income shares
- For many countries this yields $\alpha \approx 0.3 - 0.4$

QUESTIONS?

Growth sources and accounting

Growth sources and accounting

- Using our production function we can decompose the GDP growth in its three main components:
  1. Capital
  2. Labor
  3. TFP
- Assume the production function is $Y_t = A_t F(K_t, N_t)$
- Take $\log$ and derivatives w.r.t. time:

$$\frac{\dot{Y}}{Y} = \frac{\dot{A}}{A} + \epsilon_{Y,K}\frac{\dot{K}}{K} + \epsilon_{Y,N}\frac{\dot{N}}{N} \tag{10}$$

- Where $\epsilon_{Y,\cdot}$ are the elasticities of output with respect to the inputs (Capital and Labor)

Growth sources and accounting

With this in mind:

1. Get data on growth rates of GDP, Capital and Labor: $\dot{Y}/Y$, $\dot{K}/K$, and $\dot{N}/N$ (quality adjustments?)

2. Estimate $\epsilon_{Y,K}$ and $\epsilon_{Y,N}$ from historical data

3. Calculate the contributions of K and N as $\epsilon_K \times \dot{K}/K$ and $\epsilon_N \times \dot{N}/N$, respectively

4. Calculate productivity growth as the residual: $\dot{A}/A = \dot{Y}/Y - \epsilon_K \times \dot{K}/K - \epsilon_N \times \dot{N}/N$

Growth sources and accounting

Table 1: Sources of GDP growth, Canada

	1951-60	1961-70	1971-80	1981-90
GDP	**4.59%**	**5.05%**	**4.05%**	**2.65%**
Inputs	3.10%	3.05%	3.14%	2.39%
Capital	1.76%	1.67%	1.58%	1.27%
Labor	1.34%	1.39%	1.56%	1.12%
TFP	1.50%	2.00%	0.91%	0.26%

	1991-00	2001-10	2005-14
GDP	**2.86%**	**1.90%**	**1.88%**
Inputs	1.60%	1.92%	1.87%
Capital	0.94%	1.10%	1.08%
Labor	0.66%	0.83%	0.79%
TFP	1.26%	-0.02%	0.01%

Source: Penn World Tables 9.0 and own calculations.

How about cross-country comparisons? (Can we use a similar approach?)

1. Assume a Cobb-Douglas prod. function $AK^{\alpha}N^{1-\alpha}$
2. Divide by N to make it per-worker: $Y/N = y$ and $K/N = k$

$$y = Ak^{\alpha} \tag{11}$$

3. Assume $\alpha = 1/3$, and $A = 1$.
4. Use data on k^i/k^j to account for income differences across countries i and j.

If the predicted values of y^i/y^j are close to the empirical ones, then we can blame capital per worker as the culprit for GDP dispersion. (can we?)

Table 2: Income differences and TFP I

Country	y	$k^{1/3}$	A_1	Country	y	$k^{1/3}$	A_1
US	1.00	1.00	1.00	Japan	0.62	0.92	0.67
Brazil	0.26	0.71	0.36	Norway	1.28	1.10	1.17
Burundi	0.02	0.19	0.09	Qatar	1.76	1.18	1.50
Chile	0.43	0.75	0.58	South Africa	0.32	0.70	0.46
China	0.19	0.62	0.30	Spain	0.77	1.10	0.70
India	0.12	0.50	0.25	Sweden	0.77	1.02	0.76
Ireland	1.16	1.15	1.02	Switzerland	0.91	1.02	0.89
Italy	0.80	1.16	0.69	UK	0.72	1.02	0.70

Source: Penn World Tables 9.0 and own calculations (year 2014).

- With our assumptions, the model is inconsistent with the data
- Can we learn something from it ?

Table 2: Income differences and TFP I

Country	y	$k^{1/3}$	A_1	Country	y	$k^{1/3}$	A_1
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Burundi	0.02	0.19	0.09	Qatar	1.76	1.18	1.50
Chile	0.43	0.75	0.58	South Africa	0.32	0.70	0.46
China	0.19	0.62	0.30	Spain	0.77	1.10	0.70
India	0.12	0.50	0.25	Sweden	0.77	1.02	0.76
Ireland	1.16	1.15	1.02	Switzerland	0.91	1.02	0.89
Italy	0.80	1.16	0.69	UK	0.72	1.02	0.70

Source: Penn World Tables 9.0 and own calculations (year 2014).

- To identify A, we use the model, and data on y and k
- From equation (11) we get $A = y/k^{\alpha}$

Growth sources and accounting

- From the above we learn that TFP plays a role in the observed income differences between countries
- How can we interpret the results?

Income differences: India vs US

$$8 = \frac{y_{\text{US}}}{y_{\text{India}}} = \frac{A_{\text{US}}}{A_{\text{India}}} \times \frac{k_{\text{US}}^{1/3}}{k_{\text{India}}^{1/3}} = 4 \times 2$$

Note that TFP is responsible for $\approx 67\%$ of income differences and Capital per worker is responsible for $\approx 33\%$.

- Repeating for different countries results in different numbers but A is still important (average 60%).
- Is A measure of our ignorance? We can add model features to decrease its role (e.g. R&D, education, variable α)

Growth sources and accounting

- Fixing α simplifies the algebra ... visit professor Diewert 's web-site to learn about Tornqvist indexes.
- This might be true for some countries, but not for all.

Table 3: Capital share of income

Country	α	Country	α
US	0.40	Japan	0.40
Brazil	0.44	Norway	0.47
Burundi	0.39	Qatar	0.81
Chile	0.55	South Africa	0.45
China	0.43	Spain	0.42
India	0.50	Sweden	0.43
Ireland	0.51	Switzerland	0.35
Italy	0.46	UK	0.39

Source: Penn World Tables 9.0 (year 2014)

- Accounting for human capital and variable labor share we get

Table 4: Income differences and TFP II

Country	y	k^α	A_2	Country	y	k^α	A_2
US	1.00	1.00	1.00	Japan	0.94	0.92	0.71
Brazil	0.36	0.75	0.48	Norway	1.25	1.10	1.30
Burundi	0.05	0.21	0.22	Qatar	1.57	1.18	1.46
Chile	0.47	0.69	0.68	South Africa	0.66	0.70	0.53
China	0.29	0.66	0.43	Spain	1.28	1.10	0.81
India	0.18	0.47	0.39	Sweden	1.10	1.02	0.85
Ireland	1.35	1.29	1.05	Switzerland	1.08	1.02	0.96
Italy	0.98	1.32	0.74	UK	1.05	1.02	0.72

Source: Penn World Tables 9.0 and own calculations (year 2014).

Table 5: Income differences explained by TFP

Country	A_1	A_2	Country	A_1	A_2
US	-	-	Japan	58%	57%
Brazil	66%	61%	Norway	48%	49%
Burundi	69%	49%	Qatar	44%	52%
Chile	56%	50%	South Africa	61%	56%
China	67%	60%	Spain	61%	61%
India	67%	54%	Sweden	57%	57%
Ireland	53%	55%	Switzerland	54%	53%
Italy	63%	64%	UK	59%	59%

Note: A_1 accounts for the share of TFP in the GDP per-worker gap while A_2 does it in GDP per-effective-units-of-labor

Source: Penn World Tables 9.0 and own calculations (year 2014)

So what is in the TFP?

- Technology, accumulated knowledge
- Institutions and culture (Acemoglu)
- Natural resources, Climate, and Geography
- Management practices? (Bloom, Van Reenen, and others)
- Efficient allocation of resources (Hsieh-Klenow, 2009)

Since we compute it as a residual, it will be all the relevant factors in production that we don't explicitly include in our model.

- Recent studies identified **misallocations** as quantitatively important
- What is it? Easy to illustrate with our production model

Example: missallocations

- Assume two firms that have the same production function, same A, and same N
- However, firm 1 has more capital: $K_1 > K_2$
- This will imply their marginal productivities are different
- If we reallocate — from the less to the more productive— we could increase output

- An efficient allocation involves equality of all marginal products across firms (proof on the board)

Growth sources and accounting: Summary

- A simple production model allows us to account for the role of inputs in GDP growth (Growth accounting)
- A simple production model also allows to account for some fraction of income differences across countries
- Nonetheless, even after controlling for human capital, the Total Factor Productivity explains a large part
  - Responsible for at least 50% of income differences.
  - Includes many dimensions that are crucial for growth.
- Factor accumulation does not come from the sky: it requires investment and time
- Investment dynamics are "induced" by productivity, so the role of TFP might be even higher

QUESTIONS?

Growth Dynamics

Growth Dynamics: Questions

- What's the relationship between the long-run standard of living and the saving rate, population growth rate, and rate of technical progress?
- How does economic growth change over time? Will it speed up, slow down, or stabilize?
- Solow (1956) and Swan (1956)

To answer those questions assume the following:

1. The production function $F(K, N)$ has the properties described in section 4
2. The economy is closed and there's no Government: Output can be used either for consumption (C) or investment (I)
3. Constant saving rate (s)
4. Everybody work and population grow as follows: $N(t) = N_0 e^{nt}$
5. Capital depreciates at a rate δ
6. Investment converts into capital with no cost or transformation.
7. Times is continuous (makes the calculations easier)

Growth Dynamics: The Solow-Swan model

The first assumption allow us to express GDP per-capita as follow

$$
\begin{aligned}
y(t) = \frac{Y(t)}{N(t)} &= \frac{AF\left(K(t), N(t)\right)}{N(t)} \\
&= AF\left(\frac{K(t)}{N(t)}, \frac{N(t)}{N(t)} \right) AF\left(\frac{K(t)}{N(t)}, \cancel{\frac{N(t)}{N(t)}}\ 1\right) \quad \text{by constant return to scale}
\end{aligned}
$$

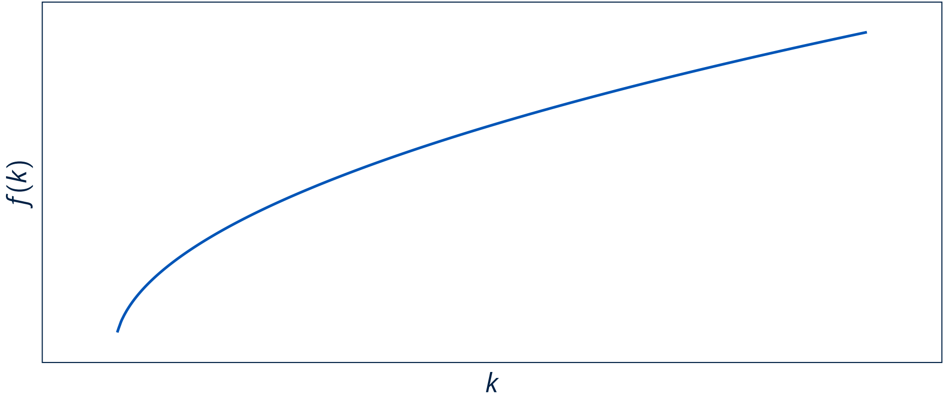
Define capital per-capita as $k(t) \equiv K(t)/N(t)$ to get

$$y(t) = AF(k(t), 1) = Af\left(k(t)\right) \tag{12}$$

where $f(\cdot) \equiv F(\cdot, 1)$ is the per-worker production function.

Note that this transformation preserves the shape of the production function

Figure 8: Per-worker Production function



- The second assumption defines the GDP expenditure:

$$Y(t) = C(t) + I(t)$$

where $C(t)$ and $I(t)$ are consumption and investment in period t, respectively. Dividing by $N(t)$ we get

$$y(t) = c(t) + i(t)$$

with $c(t) \equiv C(t)/N(t)$, and $i(t) \equiv I(t)/N(t)$

- The constant saving rate (assumption 3) implies that

$$i(t) = y(t) - c(t) = sAf\,(k(t)) \tag{13}$$

$$c(t) = y(t) - i(t) = (1-s)Af\,(k(t)) \tag{14}$$

- Assumption 4 implies that $\dot{N}/N = n$
- The last three assumptions imply that capital evolves as follow

$$\dot{K}(t) = I(t) - \delta K(t) \tag{15}$$

- The evolution of Capital per-capita is:

$$\begin{aligned}\dot{k}(t) &= \left(\dot{\frac{K(t)}{N(t)}}\right) = \frac{\dot{K}(t)N(t) - \dot{N}(t)K(t)}{N(t)^2} \\ \dot{k}(t) &= i(t) - \delta k(t) - nk(t) \quad \text{using (15) and } \dot{N}/N\end{aligned}$$

$$\dot{k}(t) = \underbrace{sAf(k(t))}_{\text{Actual investment}} - \underbrace{(\delta + n)k(t)}_{\text{break-even investment}} \quad \text{using (13)} \tag{16}$$

Remember that the derivative is taken over a division

- The solution of the model consists in a path for $k(t)$ given any initial condition $k(0)$
- Once we know $k(t)$ for all t, we can get $y(t)$, $c(t)$, and $i(t)$
- The solution can be found solving the differential equation (16), but we need to assume a functional form for $f(k)$
- We can learn more about the solution and its properties by keeping $f(k)$ as a general function and finding how the solution will look like
- For that we need to define a new concept: **Steady State**.

Steady state

Definition: Steady state

> *"In systems theory, a system or a process is in a steady state if the variables (called state variables) which define the behavior of the system or the process are unchanging in time."*
>
> — *Paul A. Gagniuc (2017)*

In our model, the system reaches steady state when $k(t)$, $y(t)$, $c(t)$, and $i(t)$ are all constants

$$\dot{y}(t) = \dot{k}(t) = \dot{c}(t) = \dot{i}(t) = 0$$

We will denote x^{ss} as $x(t)$ in steady state

Note that in steady state $K(t)$, $Y(t)$, $C(t)$ and $I(t)$ grow at rate n

By equation (16) in steady state we get

$$sAf(k^{ss}) = (\delta + n)k^{ss} \tag{17}$$

where (17) solves for k^{ss}

Steady state capital: Cobb-Douglas

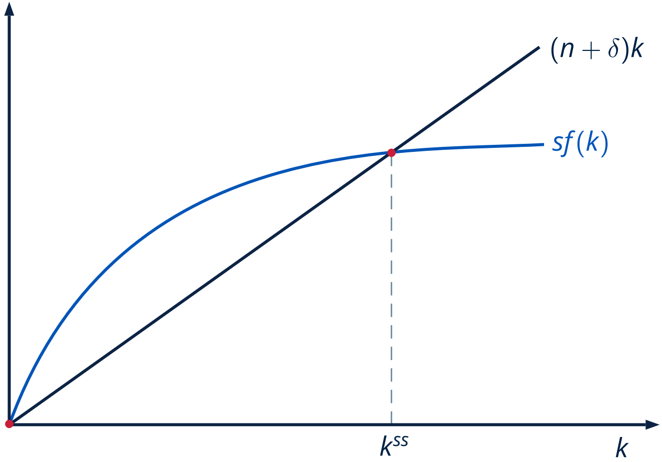
When the production function is Cobb-Douglas with capital share α, then its per-worker version is

$$f(k) = k^{\alpha}$$

Substituting this expression in (17) we get

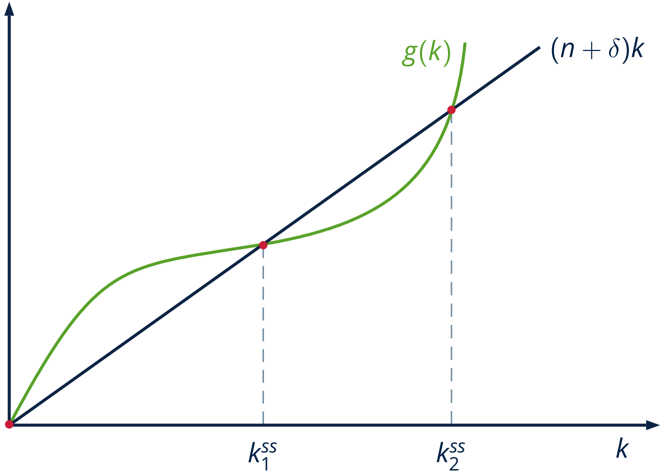
$$k^{ss} = \left(\frac{sA}{\delta + n}\right)^{1/(1-\alpha)}$$

Figure 9: Steady state in the Solow model



Steady state: uniqueness and existence

Figure 10: Steady state existence/uniqueness and assumptions on $f(k)$



Once we know k^{ss} we can find c^{ss}, i^{ss}, and y^{ss}

$$\begin{aligned} c^{ss} &= (1-s)Af\left(k^{ss}\right) = Af\left(k^{ss}\right) - (\delta + n)k^{ss} \\ i^{ss} &= sAf\left(k^{ss}\right) = (\delta + n)k^{ss} \\ y^{ss} &= Af\left(k^{ss}\right) \end{aligned}$$

Steady state: convergence

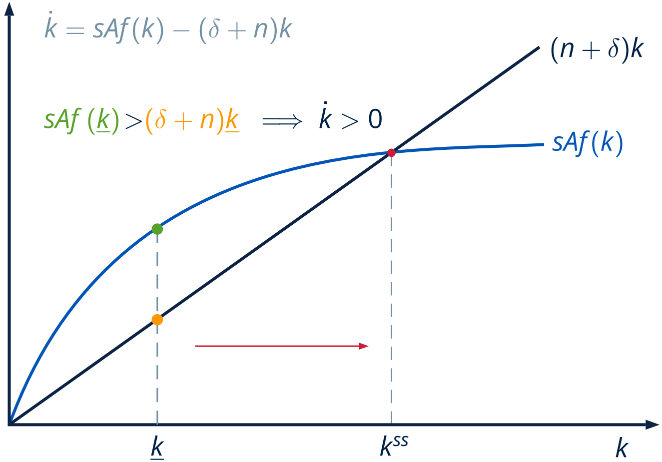
We now that once we are in steady state the economy does not change, but what happens if we are not in it? do we converge to it or not?

1. What happens when $k(t) < k^{ss}$?
   - if $k(t) < k^{ss}$ that implies $sAf(k(t)) > (\delta + n)k(t)$ (why?)
   - Then by (16) we get $\dot{k}(t) > 0$ which means that capital is increasing.
   - If $k(t)$ is increasing and $k(t) < k^{ss}$, then $k(t)$ is getting closer to k^{ss}
2. What happens when $k(t) > k^{ss}$?
   - if $k(t) > k^{ss}$ that implies $sAf(k(t)) < (\delta + n)k(t)$ (why?)
   - Then by (16) we get $\dot{k}(t) < 0$ which means that capital is decreasing.
   - If $k(t)$ is decreasing and $k(t) > k^{ss}$, then $k(t)$ is getting closer to k^{ss}

No matter where we start, the economy converges to the steady state.

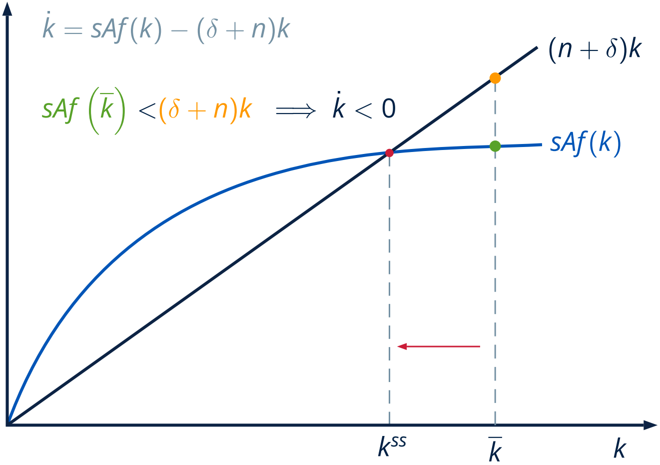
Steady state: convergence

Figure 11: Steady state convergence (case 1)



Steady state: convergence

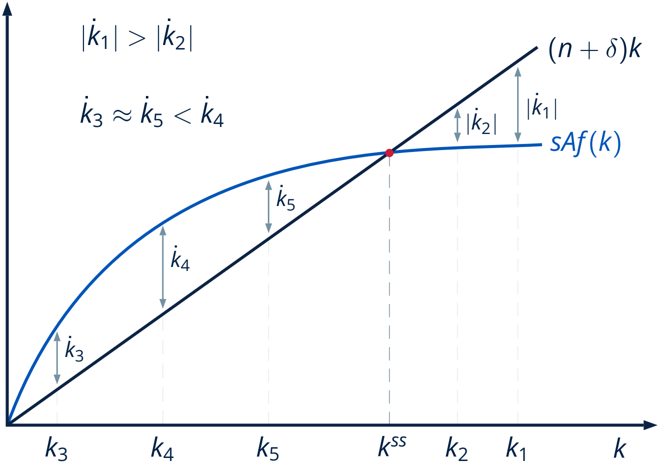
Figure 12: Steady state convergence (case 2)



Speed of convergence and capital dynamics

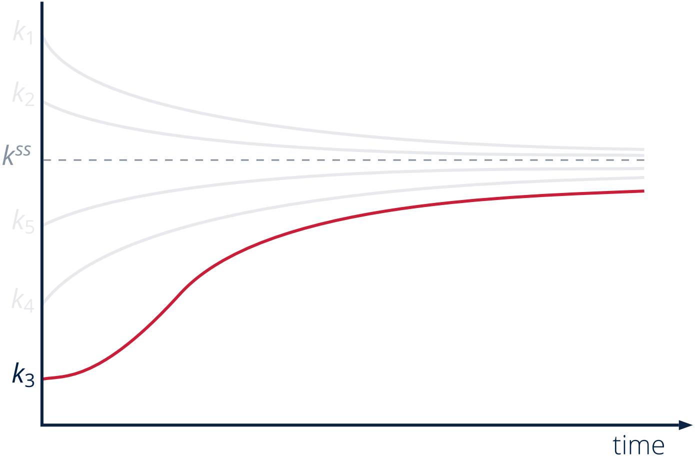
- How will the path of capital will look like while converging to the steady state?
- Close to the steady state the answer is always slowing down the closer it gets to k^{ss}. The economy will never reach the steady state, but it will get closer and closer.
- To see this note how $\dot{k}$ gets smaller and smaller when $k(t)$ approaches k^{ss}
- Exception: When $k(t)$ is close to zero, $\dot{k}$ first increase before $k(t)$ starts slowing down

Figure 13: Steady state convergence speed



Speed of convergence

Figure 14: Capital dynamic with different initial conditions

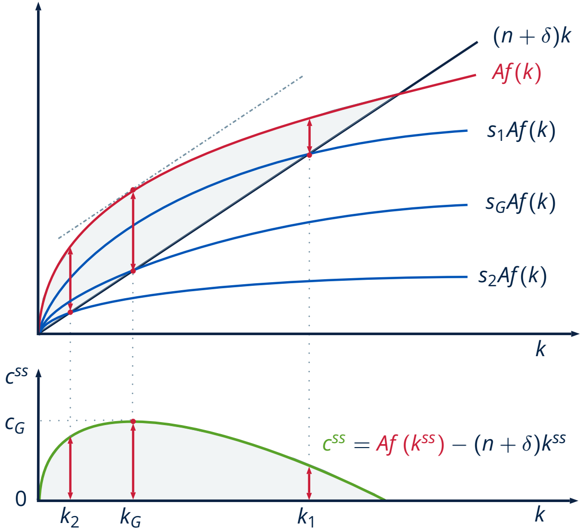


Steady state consumption

- In most models we think about utility from consumption and leisure as the measure of welfare (here, leisure is constant)
- How can we achieve more consumption in the steady state?
- How can we achieve higher or lower k^{ss}?
- What happens to c^{ss} if k^{ss} is too high?
- what happens to c^{ss} if k^{ss} is too low?

Steady state consumption: Golden Rule

Figure 15: Golden Rule level of k



Steady state consumption: Golden Rule

Golden rule level of capital

The steady state capital consistent with the maximum steady state consumption c_G is called the *golden rule level of capital*. We will denote it by k_G.

- How to find it? Look for k that maximizes the difference between production and investment in steady state:

$$\max_{k^{SS}} Af\left(k^{SS}\right) - (\delta + n)k^{SS}$$

- The first order condition of the previous problem is

$$Af'\left(k_G\right) = (\delta + n) \tag{18}$$

Golden rule level of capital: Cobb-Douglas

Replacing $f'(k) = \alpha k^{\alpha-1}$ in (18) we get

$$k_G = \left(\frac{A\alpha}{n+\delta}\right)^{1/(1-\alpha)}$$

- The properties on the production function ensure a unique steady state.
- There is a unique mapping between the level of savings and k^{ss}
- In particular, there is a unique level of savings (call it s_G) that produces k_G.
- In the Cobb-Douglas example $s_G = \alpha$

Steady state consumption: Dynamic Inefficiency

- Is it always beneficial to increase the saving rate?
- Would you save a larger part of your income knowing that you will consume less in the long run?

Dynamic Efficiency

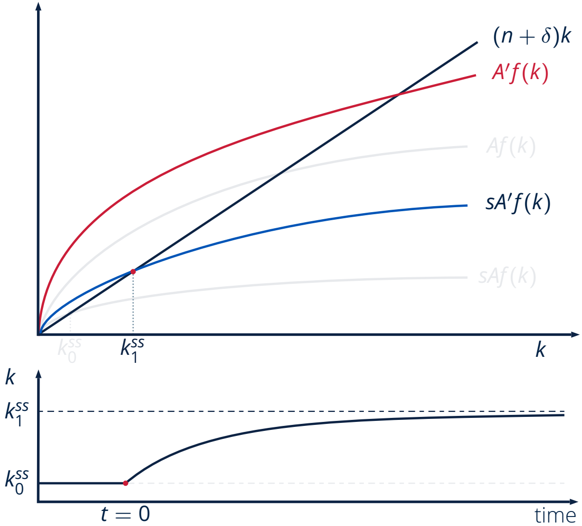
Usually there's a trade off between saving and consuming. In the short run if we save a lot, we can consume little. We would do this under the promise of consuming more in the future. Nonetheless, when $s > s_G$, not only we consume little in the short run, but the extra savings will not translate in greater consumption in steady state (long-run) either. In this case we say that the economy is **Dynamically Inefficient**.

When $s \leq s_G$ the economy is **Dynamically Efficient**, since the long-run costs *might* be offset by the short-run benefits

TFP increase

- Assume the economy is at the steady state and then A jumps to $A' > A$
- The production curve expands; with constant s, actual investment increases. $sA'f(k) > sAf(k)$
- At the current steady state, actual investment is higher than what is required to keep the current k constant. $\dot{k} > 0$
- k increases, and converges to a new, higher steady state
- Naturally, output per worker increases as well
- In addition, consumption per worker is higher at all levels of k

Figure 16: Increase in TFP

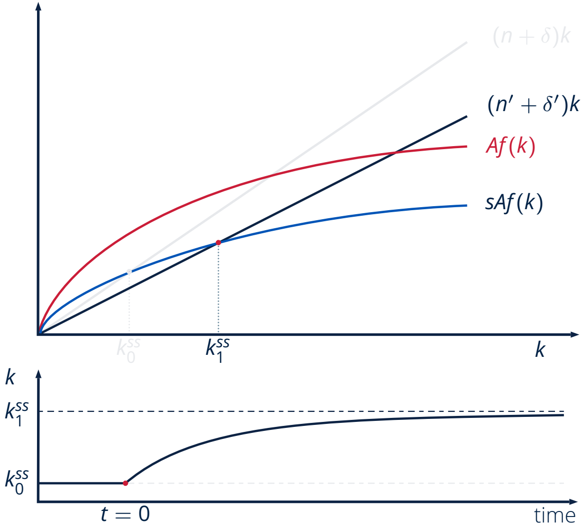


Decrease in population growth or depreciation rate

- Assume the economy is at the steady state and then $(n+\delta)$ fall to $(n'+\delta') < (n+\delta)$
- The slope of the break even investment fall and the actual investment is not affected
- At the current steady state, actual investment is higher than what is required to keep the current k constant. $\dot{k} > 0$ since $sAf(k) > (n'+\delta')k$
- k increases, and converges to a new, higher steady state
- Output per worker and consumption increase as well

Decrease in population growth or depreciation rate

Figure 17: Decrease in n or δ



Solow-Swan Model: some implications

- Are poor countries expected to catch up with rich countries?
- In the steady state all per-capita variables are constant.
- The model predicts convergence to the Steady State.
- Unconditional Convergence: all the countries have the same fundamentals $\Longrightarrow$ same steady state.
- The data suggests Conditional convergence: countries with different fundamentals will converge to different steady states.

Solow-Swan Model: some implications

- We like the Solow-Swan model steady-state convergence: the data shows a constant growth rate.
- We can include a tech process to get a **balanced growth path**.
- To be consistent with a steady state, growth can only come from labor augmenting technology. (see Barro and Sala-i Martin, 2003, ch. 1, pp. 78-80.)
- If there's a capital augmenting tech, then the production function has to be Cobb-Douglas type if a steady state exists. (see Barro and Sala-i Martin, 2003, ch. 1, pp. 78-80.)
- We will learn about labor-augmenting technology in the problem set.

Summary

- Is it reasonable to think about the saving rate as exogenous?
  (Ramsey-Cass-Koopmans model and dynamic-efficient steady state)
- The model suggests a strong link between fundamentals and the standard of living
- Some insights appear to be supported by data
- Open questions:
  - Why do fundamentals differ across countries to begin with?
  - How to promote growth? Decrease n? Increase s? Growth of TFP?
- The neoclassical production function and the Solow-Swan model cannot explain why technology growth happens.
- Development vs economic growth fields nowadays. (trade-offs)

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End

QUESTIONS?